Online Algorithms for Budget-Constrained DR-Submodular Maximization

Omid Sadeghi [1] **Reza Eghbali** [2] **Maryam Fazel** [1]

Abstract

In this paper, we study a certain class of online optimization problems, where the goal is to maximize a function that is DR-submodular (and generally non-concave) under budget constraints. We introduce a primal-dual algorithm, called the generalized sequential algorithm, and we obtain the first bound on the competitive ratio of online monotone DR-submodular function maximization subject to linear packing constraints which matches the known tight bound in the special case of the linear objective function.

1. Introduction

Online optimization covers a large number of problems including online resource allocation, online bipartite matching (Karp et al., 1990), the "Adwords" problem (Mehta et al., 2007), online submodular welfare maximization (Lehmann et al., 2006), online linear programming (Buchbinder and Naor, 2009) and online concave packing problem (Azar et al., 2016; Eghbali and Fazel, 2016). One type of algorithms proposed for solving such problems are primal-dual algorithms where the dual variable is updated at each step and is used to get the update rule for the primal variable (Buchbinder et al., 2009).

Depending on how much information about the online input is available in advance to the algorithm, online problems have been categorized into adversarial (worst-case) and stochastic input models and we consider the former in this paper. In the adversarial model, it is assumed that the algorithm has no knowledge of the online input. Performance of online algorithms is measured by their competitive ratio defined as the ratio of the value of the objective function at the output of the algorithm to the maximum objective value attained offline. In the worst-case model, one is interested in deriving lower bounds on the competitive ratio of the algorithm that holds for all arbitrary online inputs.

[1]University of Washington, Seattle, WA 98195, USA [2]Tetration Analytics, Cisco Systems, Inc., Palo Alto, CA 94301, USA. Correspondence to: Omid Sadeghi <omids@uw.edu>.

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There exists a significant amount of work on online adversarial packing problems in the literature. Buchbinder and Naor (2009) studied online packing problems with linear objective functions and Azar et al. (2016), Eghbali and Fazel (2016), and Devanur and Jain (2012) considered the more general setting with concave utility functions and obtained competitive ratio bounds in these settings. All the existing works focus on online concave packing problems. On the contrary, in this paper, we discuss a certain class of online packing problems where the objective function is DR-submodular (defined in Section 2). The DR-submodularity of a continuous function is very similar to submodularity of set functions (Bian et al., 2017b). We exploit this connection and using techniques from the submodular set function maximization literature, we introduce a greedy primal-dual algorithm, called the generalized sequential algorithm, and we analyze its performance under the worst-case input model. Specifically, we make the following contributions:

- We introduce online DR-submodular maximization subject to linear packing constraints which generalizes online packing problems by allowing the utility functions to be DR-submodular rather than concave. The generalized continuous version of online knapsack-constrained monotone submodular function maximization (Maehara et al., 2018) is a well-known example for this setting. Our framework unifies two existing separate literature on online packing problems and submodular maximization into a single rich model.
- We introduce the generalized sequential algorithm (Section 4) to solve this class of problems. The generalized sequential algorithm extends the sequential algorithm in Eghbali and Fazel (2016), however, the sequential algorithm of Eghbali and Fazel (2016) could merely be used for concave utility functions whereas our generalized sequential algorithm could be exploited for non-concave DR-submodular problems as well. Additionally, our algorithm could be interpreted as the online counterpart of the Frank-Wolfe variant proposed by Bian et al. (2017b) for solving offline constrained continuous DR-submodular maximization problems. Denoting the number of linear packing constraints by n, we consider two cases $n > 1$ and $n = 1$ separately and by designing problem-tailored penalty functions for each case to enforce the packing constraints, we derive competitive ratio bounds which are optimal in the special case of linear utility functions.

$n > 1$: In this case, our problem generalizes the Adwords problem (Mehta et al., 2007) and the online linear programming problem (Buchbinder and Naor, 2009) by allowing the utility function to be DR-submodular rather than linear. Specifically, if the objective function is linear, our problem reduces to online linear programming and if in addition to the linearity of the objective function, the coefficients in the linear packing constraint and the objective function are equal, the problem simplifies to the Adwords problem. For this setting, we obtain the first competitive ratio bound in Theorem 4.1 which is optimal in the special cases.
$n = 1$: In this case, our problem is the generalization of online knapsack-constrained monotone submodular function maximization (Maehara et al., 2018) to the continuous setting. For this online problem, we obtain a competitive ratio bound of $\frac{1}{1-\alpha+\ln(\frac{U}{L})}$ in Theorem 4.2 where L and U are lower and upper bounds on the value-to-weight ratio of the items respectively and α captures the curvature of the DR-submodular utility function (L, U and α are defined in Section 4). For discrete online knapsack-constrained submodular maximization, Maehara et al. (2018) obtained a $\frac{1}{(1+\kappa_f)(1+ln(\frac{U}{L}))}$ competitive ratio bound where κ_f is the total curvature of the submodular utility function f (Conforti and Cornuéjols, 1984). If we apply our generalized sequential algorithm to the multilinear extension of the function f (which satisfies the DR property and we denote it by F), we obtain the competitive ratio bound $\frac{1}{1-\alpha_F+\ln(\frac{U}{L})}$ and because we have showed in Remark 4.1 that $\alpha_F \geq -\kappa_f$ holds, our bound improves upon the result of Maehara et al. (2018). Additionally, if the utility function is linear, we obtain the competitive ratio bound of $\frac{1}{1+\ln(\frac{U}{L})}$ which is provably optimal (Zhou et al., 2008).
The notations used in this paper are defined in the appendix. Moreoever, proof for all the claims and results in the paper are provided in the appendix.

2. DR-submodular functions

We say that a differentiable function $F : \mathcal{X} \to \mathbb{R}$, $\mathcal{X} \subset \mathbb{R}_+^m$, is DR-submodular if its gradient is an order-reversing mapping, i.e., we have:

$$x \succeq y \Rightarrow \nabla F(x) \preceq \nabla F(y).$$

For a twice differentiable function F, it is DR-submodular if and only if its Hessian matrix $\nabla^2 F$ is entry-wise non-positive. It is noteworthy that although DR-submodularity and concavity are equivalent for the special case of $m = 1$, DR-submodular functions are generally non-concave. Nonetheless, an important consequence of DR-submodularity is concavity along non-negative directions (Bian et al., 2017b; Calinescu et al., 2011), i.e., for all x, y such that $x \preceq y$, we have $F(y) \leq F(x) + \langle \nabla F(x), y - x \rangle$. This property makes it possible to generalize previous online algorithms for concave problems to generally non-concave DR-submodular problems. We exploit this property to design our algorithm. There are many functions which satisfy the DR-submodularity property. In particular, we mention a number of them in Appendix A.

3. Problem statement

The offline constrained optimization problem is as follows:

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^n H_i(\hat{x}_i) \\ \text{subject to} & x_t \in F_t \quad \forall t \in [m] \\ & \hat{c}_i^T \hat{x}_i \leq 1 \quad \forall i \in [n], \end{array} \tag{1}$$

where $\hat{x}_i^T \in \mathbb{R}_+^m$ is the i-th row and $x_t \in \mathbb{R}_+^n$ is the t-th column of the variable matrix $X \in \mathbb{R}_+^{n\times m}$, $\hat{c}_i^T \in \mathbb{R}_+^m$ and $c_t \in \mathbb{R}_+^n$ are the i-th row and t-th column of the cost matrix $C \in \mathbb{R}_+^{n\times m}$ respectively, and $F_t \subseteq \mathbb{R}_+^n$. For all $i \in [n]$, $H_i : \mathcal{X} \to \mathbb{R}$, $\mathcal{X} \subset \mathbb{R}_+^m$, is a differentiable monotone non-decreasing DR-submodular function which is zero at the origin (i.e., $H_i(0) = 0$). For all $t \in [m]$, F_t is a compact convex constraint set that contains the origin and $\|x\|_2 \leq \lambda$ for all $x \in F_t$.
In the online setting, at step $t \in [m]$, c_t and F_t arrive online and the algorithm should choose $x_t \in F_t$ to maximize the overall objective function. Note that at each step $t \in [m]$, the function H_i $\forall i \in [n]$ is only known over subsets of variables that have already arrived. Thus, we do not have access to the objective function in advance. This setting is similar to the framework considered in streaming submodular maximization literature (Feldman et al., 2018), however, unlike streaming algorithms, the decisions of our algorithm are irrevocable.
The penalized formulation of problem (1) is the following:

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^n \big(H_i(\hat{x}_i) + G_i(\hat{c}_i^T \hat{x}_i)\big) \\ \text{subject to} & x_t \in F_t \subseteq \mathbb{R}_+^n \ \forall t \in [m] \end{array} . \tag{2}$$

As an example, if for all $i \in [n]$,

$$G_i(u) = \begin{cases} 0 & \text{if } 0 \leq u \leq 1 \\ -\infty & \text{if } u > 1 \end{cases},$$

i.e., the concave indicator function of the interval $[0, 1]$, the above two optimization problems are equivalent.
We aim to design differentiable, concave and monotone non-increasing penalty functions $G_i : \mathbb{R}_+ \to \mathbb{R}$ $\forall i \in [n]$ and use them in our online algorithm such that the output does not violate any of the linear packing constraints.
Karush–Kuhn–Tucker (KKT) conditions for the penalized problem (2) can be written as: (see Appendix B for the

derivation of the dual problem.)

$$x_t^* \in \arg\max_{x\in F_t} \langle x, \begin{bmatrix} y_{1t}^* - z_1^* c_{1t} \\ \vdots \\ y_{nt}^* - z_n^* c_{nt} \end{bmatrix} \rangle,$$
$$\hat{y}_i^* = \nabla H_i(\hat{x}_i^*) \; i = 1, \ldots, n,$$
$$z_i^* = -G_i'(\hat{c}_i^T \hat{x}_i^*) \; i = 1, \ldots, n,$$

where G_i' is the derivative of the scalar penalty function G_i. We remind the reader that we aim to design differentiable penalty functions $G_i \; \forall i \in [n]$ and therefore, we have used G_i' in the KKT conditions. We will use these KKT conditions to design the generalized sequential algorithm.

4. Generalized sequential algorithm and competitive ratio analysis

In this section, we first introduce our design for the penalty functions in Section 4.1. Then, we propose our algorithm, called generalized sequential algorithm, in Section 4.2 and finally, all the competitive ratio bounds are presented in Section 4.3.

4.1. Design of the penalty functions

If $n = 1$, we design the penalty function G_1 as follows:

$$G_1(u) = \begin{cases} -L_1 u & \text{if } 0 \le u < \frac{1}{\ln\left(\frac{U_1 e}{L_1}\right)} \\ -\frac{1}{\ln\left(\frac{U_1 e}{L_1}\right)} \frac{L_1}{e} \left(\frac{U_1 e}{L_1}\right)^u & \text{if } u \ge \frac{1}{\ln\left(\frac{U_1 e}{L_1}\right)} \end{cases}.$$

If $n > 1$, for all $i \in [n]$, we construct the penalty function G_i as below:

$$G_i(u) = \frac{L_i}{(e-1)\ln(1 + \frac{U_i(e-1)}{L_i})}\left(1 - (1 + \frac{U_i(e-1)}{L_i})^u\right) + \frac{L_i}{e-1} u,$$

where for all $i \in [n]$, U_i and L_i are defined as follows:

$$U_i = \max_{t\in[m]} \frac{\sup_{x\in\mathbb{R}^m:\, \hat{c}_i^T x = 1} \nabla_t H_i(x)}{c_{it}},$$
$$L_i = \min_{t\in[m]} \frac{\inf_{x\in\mathbb{R}^m:\, \hat{c}_i^T x \le 1} \nabla_t H_i(x)}{c_{it}}.$$

Roughly speaking, U_i and L_i are upper and lower bounds for the value-to-weight ratio of the items for the i-th agent respectively. We are assuming that these upper and lower bounds are available offline to design the penalty functions. Our design for the penalty function for $n = 1$ is inspired by the threshold function proposed by Zhou et al. (2008). In the $n > 1$ case, our penalty functions are inspired by the allocation rule of the primal-dual algorithm for the Adwords problem (Mehta et al., 2007).

4.2. Generalized sequential algorithm

The generalized sequential algorithm is presented in Algorithm 1. The algorithm outputs $\tilde{x}_t$ at each online step $t \in [m]$.
For all $i \in [n]$, $t \in [m]$ and $k \in \{0, \ldots, K\}$, define:

$$\omega_{it}(k) := \big[[\tilde{x}_1]_i, \ldots, [\tilde{x}_{t-1}]_i, [\tilde{x}_t(k)]_i, \underbrace{0, \ldots, 0}_{m-t \text{ times}}\big]^T,$$
$$[d_t(k)]_i := \nabla_t H_i\big(\omega_{it}(k)\big) + c_{it} G_i'\big(\hat{c}_i^T \omega_{it}(k)\big). \tag{3}$$

In the above definitions, we have used the notation $[u]_i$ to denote the i-th entry of the vector u and ∇_t denotes the t-th entry of the gradient vector.
$\tilde{x}_t$ is the convex combination (average) of vectors in the convex set F_t and hence, $\tilde{x}_t \in F_t$ holds. At each online step $t \in [m]$, the algorithm performs a total of K Frank-Wolfe updates in its inner loop where in each of these updates, a linear maximization problem over the set F_t is solved. Note that in our applications, F_t is usually a box constraint or the simplex and therefore, the corresponding linear maximization problem could be solved efficiently. In order to guarantee none of the budget constraints are violated, the design is such that all penalty functions are concave and monotone non-increasing, and for all $i \in [n]$, $G_i'(1) = -U_i$ holds. Therefore, for all u such that $\hat{c}_i^T u \ge 1$, $\nabla_t H_i(u) + c_{it} G_i'(\hat{c}_i^T u) \le 0$ holds for all $t \in [m]$, and considering that $0 \in F_t$, the algorithm would not assign more items to the i-th agent.
At step $t \in [m], k \in [K]$, the update rule for $v_t(k)$ has the same form as the KKT condition for x_t^*. In other words, the generalized sequential algorithm uses $\nabla H_i(\omega_{it}(k-1))$ and $-G_i'(\hat{c}_i^T \omega_{it}(k-1))$ as the current estimate of $\hat{y}_i^*$ and z_i^* respectively and using them, the algorithm obtains $v_t(k)$ to improve the estimate of x_t^*.

Algorithm 1 Generalized sequential algorithm

Input: Penalty functions $G_i \; \forall i \in [n]$ and K.
Initialize $\tilde{X} = 0$.
for $t = 1$ **to** m **do**
  c_t, F_t arrive online and gradient of $H_i \; \forall i \in [n]$ over the first t variables (i.e., all other $m - t$ variables being zero) is accessible.
  $\tilde{x}_t(0) = 0$.
  for $k = 1$ **to** K **do**
    $v_t(k) = \arg\max_{v\in F_t} \langle v, d_t(k-1)\rangle$.
    $\{d_t(k-1)$ defined in equation (3)$\}$
    $\tilde{x}_t(k) = \tilde{x}_t(k-1) + \frac{1}{K} v_t(k)$.
  end for
  Output: $\tilde{x}_t = \tilde{x}_t(K)$.
end for

We define:

$$\text{ALG} := \sum_{i=1}^{n} H_i(\omega_{im}(K)),$$
$$\text{P}_{\text{gseq}} := \sum_{i=1}^{n} \big(H_i(\omega_{im}(K)) + G_i(\hat{c}_i^T \omega_{im}(K))\big).$$

ALG and P_{gseq} are the objective values of problems (1) and (2) at the end of the algorithm respectively.

4.3. Competitive ratio analysis

First, we remind the reader that $H_i\ \forall i \in [n]$ are DR-submodular and not necessarily concave. On the other hand, $G_i\ \forall i \in [n]$ are the designed concave penalty functions. In order to derive the competitive ratio, we make the following smoothness assumption about the functions:

Assumption 1: For all $i \in [n]$, functions H_i and G_i have an L-Lipschitz gradient, i.e., for all $x \in \mathcal{X}$ and $u \in \mathbb{R}^m$ where $u \succeq 0$ or $u \preceq 0$, the following holds:

$$H_i(x+u) - H_i(x) \geq \langle u, \nabla H_i(x)\rangle - \frac{L}{2}\|u\|^2.$$

Also, for all $x \in \mathbb{R}$ and $v \in \mathbb{R}$, we have:

$$G_i(x+v) - G_i(x) \geq vG_i'(x) - \frac{L}{2}v^2.$$

We also define the parameter α as follows:

Definition 4.1. *For all $i \in [n]$, α_{H_i} is defined as:*

$$\begin{aligned}\alpha_{H_i} &:= \sup\{\beta \mid H_i^*\big(\nabla H_i(u)\big) \geq \beta H_i(u)\ \forall u : \hat{c}_i^T u \leq 1\}\\ &= \sup\{\beta \mid \langle \nabla H_i(u), u\rangle \geq (1+\beta)H_i(u)\ \forall u : \hat{c}_i^T u \leq 1\}\\ &= \inf_{u:\hat{c}_i^T u \leq 1} \frac{\langle \nabla H_i(u), u\rangle}{H_i(u)} - 1.\end{aligned}$$

Since H_i is monotone non-decreasing, $0 \leq \langle \nabla H_i(u), u\rangle$ holds. Additionally, because H_i is DR-submodular and $H_i(0) = 0$, we have $\langle \nabla H_i(u), u\rangle \leq H_i(u)$. Thus, $-1 \leq \alpha_{H_i} \leq 0$ always holds.

The parameter α characterizes the curvature of the function over the domain of the algorithm (i.e., where the budget constraint is not violated). For linear functions F, $\alpha_F = 0$ and larger $|\alpha_F|$ corresponds to the function F being more curved/non-linear. In fact, α of the multilinear extension of a discrete submodular function and the total curvature of the underlying submodular set function are related as follows:

Remark 4.1. ***Connection between total curvature of a submodular function and α:***

Recall that for a non-negative normalized monotone non-decreasing submodular function $f : 2^V \to \mathbb{R}$, total curvature is defined as: (Conforti and Cornuéjols, 1984)

$$\kappa_f = 1 - \min_{j: f(j)\neq 0} \frac{f(j|(V\setminus j))}{f(j)} = 1 - \min_{S\subset V\setminus\{j\}, f(j)\neq 0} \frac{f(j|S)}{f(j)}.$$

If we denote the multilinear extension of this function by F, we have $\alpha_F \geq -\kappa_f$.

If we denote the optimal values of the original constrained problem (1) and its dual problem by OPT and D^* respectively, $\text{OPT} \leq \text{D}^*$ holds due to weak duality.

We state our main results, i.e., competitive ratio bounds, below.

Theorem 4.1. *For $n > 1$, if Assumption 1 holds, then for the generalized sequential algorithm, we have:*

$$\frac{\text{ALG}}{\text{OPT}} \geq \frac{\text{ALG}}{\text{D}^*} \geq \frac{1 - \frac{1}{\text{D}^*}\max_{i\in[n]} \gamma_i \frac{Lm\lambda^2}{K}}{-\min_{i\in[n]} \alpha_{H_i} + \frac{e}{e-1}\max_{i\in[n]} \gamma_i},$$

where $\gamma_i := \ln(1 + \frac{U_i(e-1)}{L_i})\ \forall i \in [n]$.

Therefore, when $K \to \infty$, the competitive ratio bound is derived as $\frac{\text{ALG}}{\text{OPT}} \geq \frac{1}{-\min_{i\in[n]}\alpha_{H_i} + \frac{e}{e-1}\max_{i\in[n]}\gamma_i}$.

The bound in Theorem 4.1 is tight in several known special cases. For the Adwords problem, since $U_i = L_i = 1$ and $\alpha_{H_i} = 0$ for all $i \in [n]$, competitive ratio of $1 - \frac{1}{e}$ is obtained which is optimal (Mehta et al., 2007). Additionally, for the online linear programming problem, considering that $\alpha_{H_i} = 0\ \forall i \in [n]$, we obtain $\big(\max_{i\in[n]} \ln(1 + \frac{U_i(e-1)}{L_i})\big)^{-1} \times (1 - \frac{1}{e})$ as the competitive ratio bound which is known to be optimal (Buchbinder and Naor, 2009).

Theorem 4.2. *For $n = 1$, if Assumption 1 holds, then for the generalized sequential algorithm, we have:*

$$\frac{\text{ALG}}{\text{OPT}} \geq \frac{\text{ALG}}{\text{D}^*} \geq \frac{1 - \frac{1}{\text{D}^*}\ln(\frac{U_1 e}{L_1})\frac{Lm\lambda^2}{K}}{1 - \alpha_{H_1} + \ln(\frac{U_1}{L_1})}.$$

Therefore, when $K \to \infty$, the competitive ratio bound is derived as $\frac{1}{1-\alpha_{H_1}+\ln(\frac{U_1}{L_1})}$.

If we use the result of Theorem 4.2 for the online linear knapsack problem (where $\alpha_{H_1} = 0$), we obtain the competitive ratio bound of $\frac{1}{1+\ln(\frac{U_1}{L_1})}$ which is optimal (Zhou et al., 2008) (note that because we allow fractional assignments, unlike Zhou et al., 2008, we do not need the $c_{1t} \ll 1\ \forall t \in [m]$ assumption to obtain the optimal competitive ratio).

Theorems 4.1 and 4.2 provide the first competitive ratio bounds that generalize the results of Azar et al. (2016) and Eghbali and Fazel (2016) for the concave case to general DR-submodular objective functions.

5. Conclusion

In this paper, we considered a class of online optimization problems, where the objective function is monotone DR-submodular under linear packing constraints. We proposed the generalized sequential algorithm for solving such problems and we obtained competitive ratio bounds for this algorithm.

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Appendices

Notations. We use $[m]$ to denote the set $\{1, 2, \ldots, m\}$. For a matrix $A \in \mathbb{R}^{n\times m}$, we denote its (i,t)-th entry by a_{it}, its i-th row by $\hat{a}_i^T$ for all $i \in [n]$, and its t-th column by a_t for all $t \in [m]$. A^T denotes the transpose of A. The inner product of two vectors $x, y \in \mathbb{R}^m$ is denoted by $\langle x, y\rangle$ or $x^T y$. For two vectors $x, y \in \mathbb{R}^m$, $x \preceq y$ means $x_i \le y_i \ \forall i \in [m]$. A function $F : \mathbb{R}^m \to \mathbb{R}$ is called monotone if for all x, y such that $x \preceq y$, $F(x) \le F(y)$ holds. We use F^* to denote the concave conjugate of a function $F : \mathbb{R}^m \to \mathbb{R}$, defined as:

$$F^*(y) = \inf_x \big(\langle x, y\rangle - F(x)\big).$$

For a convex set $\mathcal{P}$, the support function of $\mathcal{P}$ is defined as:

$$\sigma_{\mathcal{P}}(x) = \sup_{y\in\mathcal{P}} \langle x, y\rangle.$$

A. Examples of continuous non-concave DR-submodular functions

Continuous extension of submodular set functions. A discrete function $f : \{0,1\}^V \to \mathbb{R}$ over the ground set V is submodular if for all $j \in V$ and $A \subseteq B \subseteq V \setminus \{j\}$, the following holds:

$$f(A \cup \{j\}) - f(A) \ge f(B \cup \{j\}) - f(B).$$

DR-submodularity is the continuous counterpart of the submodularity property of set functions (Bian et al., 2017b). Indeed, the multilinear extension (Calinescu et al., 2007) and the softmax extension (Gillenwater et al., 2012) of submodular set functions are DR-submodular.

Indefinite quadratic functions. Let $F(x) = \frac{1}{2}x^T P x + p^T x + c$. If the matrix P is element-wise non-positive, F is a DR-submodular function.

Concave functions with negative dependence. If $H_i : \mathbb{R} \to \mathbb{R}$ is concave for all $i \in [m]$ and $\theta_{ij} \le 0 \ \forall i \ne j$, the following function $F : \mathbb{R}_+^m \to \mathbb{R}$ is DR-submodular:

$$F(x) = \sum_{i=1}^m H_i(x_i) + \sum_{i,j: i\ne j} \theta_{ij} x_i x_j.$$

Note that indefinite quadratic functions are a special example of the above where all the concave functions H_i are quadratic with negative coefficients.

Log-determinant function. Let the function $F : [0,1]^m \to \mathbb{R}$ be defined as

$$F(x) = \log\det\big(\mathrm{diag}(x)(L - I) + I\big),$$

where $L \succeq 0$ is a positive semidefinite matrix and $\mathrm{diag}(x)$ denotes a diagonal matrix with vector x on its diagonal. This function is extensively used as the utility function in Determinantal Point Processes (DPPs). It was proved in (Gillenwater et al., 2012) that F is a DR-submodular function. In fact, F is the softmax extension of the submodular set function $\log\det(L_S)$ over the ground set V where L_S is the submatrix of L whose rows and columns are characterized by the set $S \subseteq V$.

See Bian et al. (2017a; 2017b) for more examples of continuous DR-submodular objective functions.

B. Derivation of the dual problem

Let

$$I_{F_t}(x) = \begin{cases} 0 & \text{if } x \in F_t \\ \infty & \text{o.w.} \end{cases},$$

i.e., the convex indicator function of the set F_t.

Remember the offline constrained optimization problem:

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^n H_i(\hat{x}_i) \\ \text{subject to} & x_t \in F_t \subseteq \mathbb{R}_+^n \ \forall t \in [m] \\ & \hat{c}_i^T \hat{x}_i \le 1 \ \forall i \in [n] \end{array} . \tag{4}$$

We can equivalently write the optimization problem as follows:

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^n H_i(\hat{d}_i) \\ \text{subject to} & x_t \in F_t \subseteq \mathbb{R}_+^n \ \forall t \in [m] \\ & \hat{c}_i^T \hat{x}_i \le 1 \ \forall i \in [n] \\ & \hat{d}_i = \hat{x}_i \ \forall i \in [n] \end{array} . \tag{5}$$

We derive the dual of problem (5) below:

$$\begin{aligned} g(\hat{y}_i, z_i) &= \inf_{\hat{d}_i, \hat{e}_i, X} \sum_{i=1}^n -H_i(\hat{d}_i) + \sum_{i=1}^n \hat{y}_i^T (\hat{d}_i - \begin{bmatrix} x_{i1} \\ \vdots \\ x_{im} \end{bmatrix}) \\ &\quad + \sum_{i=1}^n z_i \big(\hat{c}_i^T \begin{bmatrix} x_{i1} \\ \vdots \\ x_{im} \end{bmatrix} - 1\big) + \sum_{t=1}^m I_{F_t}(x_t) \\ &= \sum_{i=1}^n \inf_{\hat{d}_i} \big(\hat{y}_i^T \hat{d}_i - H_i(\hat{d}_i)\big) - \sum_{i=1}^n z_i \\ &\quad + \sum_{t=1}^m \inf_{x_t \in F_t} \big(I_{F_t}(x_t) - \langle \underbrace{\begin{bmatrix} y_{1t} - z_1 c_{1t} \\ \vdots \\ y_{nt} - z_n c_{nt} \end{bmatrix}}_{v_t}, x_t\rangle\big) \\ &= \sum_{i=1}^n H_i^*(\hat{y}_i) - \sum_{i=1}^n z_i - \sum_{t=1}^m \sup_{x_t \in F_t} \big(\langle v_t, x_t\rangle - I_{F_t}(x_t)\big) \\ &= \sum_{i=1}^n H_i^*(\hat{y}_i) - \sum_{i=1}^n z_i - \sum_{t=1}^m \sigma_{F_t}(v_t), \end{aligned}$$

where $\hat{y}_i = [y_{i1}, \ldots, y_{im}]^T$, $\sigma_{F_t}(u) = \sup_{w\in F_t} w^T u$ is the support function of the set F_t and $H_i^*(u) = \inf_w \big(w^T u -$

$H_i(w)\big)$ is the concave conjugate function of H_i. Therefore, the dual problem is:

$$\begin{array}{ll} \text{minimize} & \sum_{t=1}^m \sigma_{F_t}\big(v_t\big) - \sum_{i=1}^n H_i^*(\hat{y}_i) + \sum_{i=1}^n z_i \\ \text{subject to} & z_i \geq 0 \; \forall i \in [n] \end{array}.$$

C. Connection between total curvature of a submodular function and α

First, note that F, i.e., the multilinear extension of the discrete submodular function f, satisfies the DR property and $F(0) = 0$. Since F is linear in each of its arguments, we can write:

$$\langle \nabla F(x), x \rangle = \sum_t \mathbb{E}_{S\sim x}[f(S \cup \{t\}) - f(S \setminus \{t\})]x_t. \quad (6)$$

Depending on whether $t \in S$ or not, one of the terms $\big(f(S \cup \{t\}) - f(S)\big)$ or $\big(f(S) - f(S \setminus \{t\})\big)$ would be zero. So, by definition of total curvature of f, i.e., κ_f, we have:

$$\begin{aligned} f(S \cup \{t\}) - f(S \setminus \{t\}) &= \big(f(S \cup \{t\}) - f(S)\big) \\ &\quad + \big(f(S) - f(S \setminus \{t\})\big) \\ &\geq (1 - \kappa_f) f(\{t\}). \end{aligned} \quad (7)$$

Combining (6) and (7), we have:

$$\langle \nabla F(x), x \rangle \geq (1 - \kappa_f) \sum_t x_t f(\{t\}). \quad (8)$$

Defining $\hat{x}_t = [x_1, \ldots, x_t, 0, \ldots, 0]^T$, we can write:

$$\begin{aligned} F(x) &= \sum_t \big(F(\hat{x}_t) - F(\hat{x}_{t-1})\big) \\ &= \sum_t x_t \nabla_t F(\hat{x}_{t-1}). \end{aligned}$$

Since $f(\{t\}) = F(1_t) = F(1_t) - F(0) = \nabla_t F(0)$, using the DR property of the function F, $\nabla_t F(0) \geq \nabla_t F(\hat{x}_{t-1})$ and therefore, we have:

$$F(x) \leq \sum_t x_t f(\{t\}). \quad (9)$$

Combining (8) and (9), we conclude:

$$\begin{aligned} \langle \nabla F(x), x \rangle &\geq (1 - \kappa_f) F(x) \\ \frac{\langle \nabla F(x), x \rangle}{F(x)} &\geq (1 - \kappa_f) \\ \alpha_F &\geq -\kappa_f. \end{aligned}$$

As a corollary, since $\alpha_F \in [-1, 0]$ and $\kappa_f \in [0, 1]$, if $\kappa_f = 0$ (i.e., f is modular), we can conclude that $\alpha_f = 0$ as well.

Example C.1. *Consider the Ising model with nonpositive pairwise interactions (Bian et al., 2019):*

$$f(v) = \sum_{i \in [m]} \theta_i v_i + \sum_{i<j} \theta_{ij} v_i v_j.$$

where $v \in \{0,1\}^{\mathcal{V}}$, $|V| = m$, $\theta_i \geq 0$, $\theta_{ij} \leq 0$ and $\theta_i \geq \sum_{j:\, j<i} |\theta_{ji}| + \sum_{j:\, j>i} |\theta_{ij}|$ for all $i, j \in [m]$ such that $i \neq j$. Under these assumptions, this class of functions are monotone submodular.
The multilinear extension of this function could be easily derived as follows:

$$F(x) = \sum_{i \in n} \theta_i x_i + \sum_{i<j} \theta_{ij} x_i x_j.$$

The gradient of this function could be written in the following way:

$$\nabla_i F(x) = \theta_i + \sum_{j:\, j<i} \theta_{ji} x_j + \sum_{j:\, j>i} \theta_{ij} x_j \;\forall i \in [n].$$

As a simple example, consider $f(v) = 2v_1 + v_2 + 2v_3 - v_1 v_3$, its multilinear extension $F(x) = 2x_1 + x_2 + 2x_3 - x_1 x_3$ and assume its corresponding linear packing constraint to be $\mathcal{P} = \{x \in [0,1]^3 :\ 0.5x_1 + 0.6x_2 + 0.75x_3 \leq 1\}$.
For κ_f, we have:

$$\begin{aligned} \kappa_f &= 1 - \min_{j: f(j) \neq 0} \frac{f(j|(V \setminus j))}{f(j)} \\ &= 1 - \min\{\frac{1}{2}, \frac{1}{1}, \frac{1}{2}\} \\ &= \frac{1}{2}. \end{aligned}$$

α_F could be computed as follows:

$$\begin{aligned} \alpha_F &= \inf_{x \in \mathcal{P}} \frac{\langle \nabla F(x), x \rangle}{F(x)} - 1 \\ &= \inf_{x \in \mathcal{P}} \frac{2x_1 + x_2 + 2x_3 - 2x_1 x_3}{2x_1 + x_2 + 2x_3 - x_1 x_3} - 1 \\ &= \inf_{x \in \mathcal{P}} \frac{-x_1 x_3}{2x_1 + x_2 + 2x_3 - x_1 x_3} \\ &= \frac{-1}{4}. \end{aligned}$$

D. Lemma D.1

Lemma D.1. *The following inequality holds for the output of the generalized sequential algorithm:*

$$\text{ALG} \geq \sum_{t=1}^m \sigma_{F_t}(d_t(K)) - \sum_{i=1}^n G_i(\hat{c}_i^T \omega_{im}(K)) - \frac{Lm\lambda^2}{K}. \quad (10)$$

Proof. Considering that $\|x\|_2 \leq \lambda$ holds for all $x \in F_t$ and $t \in [m]$, we can write:

$$P_{\text{gseq}} = \sum_{i=1}^n \big(H_i(\omega_{im}(K)) + G_i((\hat{c}_i^T \omega_{im}(K))\big)$$

$$= \sum_{i=1}^n \sum_{t=1}^m \big((H_i(\omega_{it}(K)) - H_i(\omega_{it}(0))\big)$$

$$
\begin{aligned}
&+\sum_{i=1}^{n}\sum_{t=1}^{m}\big(G_i(\hat{c}_i^T\omega_{it}(K))-G_i\hat{c}_i^T\omega_{it}(0))\big)\\
&=\sum_{i=1}^{n}\sum_{t=1}^{m}\sum_{k=1}^{K}\big((H_i(\omega_{it}(k))-H_i(\omega_{it}(k-1))\big)\\
&+\sum_{i=1}^{n}\sum_{t=1}^{m}\sum_{k=1}^{K}\big(G_i((\hat{c}_i^T\omega_{it}(k))-G_i(\hat{c}_i^T\omega_{it}(k-1)))\big)\\
&\overset{(a)}{\geq}\sum_{i=1}^{n}\sum_{t=1}^{m}\sum_{k=1}^{K}\big(\frac{1}{K}[v_t(k)]_i\nabla_t H_i(\omega_{it}(k-1))-\frac{L}{2K^2}[v_t(k)]_i^2\big)\\
&+\sum_{i=1}^{n}\sum_{t=1}^{m}\sum_{k=1}^{K}\big(\frac{1}{K}c_{it}[v_t(k)]_iG_i'(\hat{c}_i^T\omega_{it}(k-1))-\frac{Lc_{it}^2}{2K^2}[v_t(k)]_i^2\big)\\
&\overset{(b)}{\geq}\sum_{i=1}^{n}\sum_{t=1}^{m}\sum_{k=1}^{K}\big(\frac{1}{K}[v_t(k)]_i\nabla_t H_i(\omega_{it}(k-1))\big)-\frac{Lm\lambda^2}{2K}\\
&+\sum_{i=1}^{n}\sum_{t=1}^{m}\sum_{k=1}^{K}\big(\frac{1}{K}c_{it}[v_t(k)]_iG_i'(\hat{c}_i^T\omega_{it}(k-1))\big)-\frac{Lm\lambda^2}{2K}\\
&=\sum_{t=1}^{m}\sum_{k=1}^{K}\frac{1}{K}\langle v_t(k),d_t(k-1)\rangle-\frac{Lm\lambda^2}{K}\\
&\overset{(c)}{=}\sum_{t=1}^{m}\sum_{k=1}^{K}\frac{1}{K}\sigma_{F_t}(d_t(k-1))-\frac{Lm\lambda^2}{K}\\
&\overset{(d)}{\geq}\sum_{t=1}^{m}\sigma_{F_t}(\frac{1}{K}\sum_{k=1}^{K}d_t(k-1))-\frac{Lm\lambda^2}{K},
\end{aligned}
$$

where (a) is due to Assumption 1, (b) follows from $\|x\|_2 \leq \lambda\ \forall x \in F_t, t \in [m]$, (c) uses the update rule of the generalized sequential algorithm and (d) is a result of subadditivity of the support function σ_{F_t}.

By definition, for all $t \in [m], i \in [n]$ and $k \in \{0,\ldots,K\}$, we have $[d_t(k)]_i := \nabla_t H_i\big(\omega_{it}(k)\big) + c_{it}G_i'\big(\hat{c}_i^T\omega_{it}(k)\big)$. Since H_i is a continuous DR-submodular function, and that $\omega_{it}(k-1) \preceq \omega_{it}(K)$ holds, $\nabla_t H_i\big(\omega_{it}(K)\big) \leq \nabla_t H_i\big(\omega_{it}(k-1)\big)$ follows. Additionally, $G_i'\big(\hat{c}_i^T\omega_{it}(K)\big) \leq G_i'\big(\hat{c}_i^T\omega_{it}(k-1)\big)$ holds due to concavity of the penalty function G_i (equivalently, G_i' being non-increasing). Combining these results, we have:

$$
\begin{aligned}
d_t(K) &\preceq d_t(k-1)\\
d_t(K) &\preceq \frac{1}{K}\sum_{k=1}^{K}d_t(k-1)\\
x^T d_t(K) &\leq \frac{1}{K}x^T\sum_{k=1}^{K}d_t(k-1),
\end{aligned}
\tag{11}
$$

for $x \in F_t \subset \mathbb{R}_+^n$, i.e., x being element-wise non-negative. Taking supremum of (11) over all $x \in F_t$, we obtain:

$$\sigma_{F_t}(d_t(K)) \leq \sigma_{F_t}(\frac{1}{K}\sum_{k=1}^{K}d_t(k-1)).$$

Therefore, we have:

$$\text{ALG} \geq \sum_{t=1}^{m}\sigma_{F_t}(d_t(K)) - \sum_{i=1}^{n}G_i(\hat{c}_i^T\omega_{im}(K)) - \frac{Lm\lambda^2}{K}.$$

□

E. Lemma E.1

Lemma E.1. *For all $i \in [n]$, we have:*

$$L_i \leq \frac{H_i\big(\omega_{im}(K)\big)}{\hat{c}_i^T\omega_{im}(K)}. \tag{12}$$

Proof. First, note that by definition $\omega_{it}(0) = \omega_{i,t-1}(K)$ and $\omega_{i1}(0) = 0$ holds for all $t \in [m], i \in [n]$. Additionally, $H_i(0) = 0$ holds by assumption for all $i \in [n]$. Therefore, using the mean-value theorem, we can write:

$$
\begin{aligned}
H_i(\omega_{im}(K)) &= \sum_{t=1}^{m}\big(H_i(\omega_{it}(K)) - H_i(\omega_{it}(0))\big)\\
&= \sum_{t=1}^{m}\tilde{x}_{it}\nabla_t H_i(u_t),
\end{aligned}
$$

where $\tilde{x}_{it} = [\tilde{x}_t]_i = [\tilde{x}_t(K)]_i$, $u_t \in \mathbb{R}^m$ is the intermediate point in the mean-value theorem and $\omega_{it}(0) \preceq u_t \preceq \omega_{it}(K)$. Thus, we can write:

$$
\begin{aligned}
L_i &\leq \min_{t\in[m]}\frac{\nabla_t H_i(u_t)}{c_{it}} \leq \frac{H_i\big(\omega_{im}(K)\big)}{\hat{c}_i^T\omega_{im}(K)} = \frac{\sum_{t=1}^{m}\tilde{x}_{it}\nabla_t H_i(u_t)}{\sum_{t=1}^{m}c_{it}\tilde{x}_{it}}\\
&\leq \max_{t\in[m]}\frac{\nabla_t H_i(u_t)}{c_{it}}.
\end{aligned}
$$

□

F. Proof of Theorem 4.1

we can use the definition of α to obtain:

$$H_i^*\big(\nabla H_i(\omega_{im}(K))\big) \geq \alpha_{H_i}H_i(\omega_{im}(K))\ \forall i \in [n]. \tag{13}$$

For all $i \in [n]$, using the definition of G_i for the $n > 1$ case and defining $\gamma_i := \ln(1 + \frac{U_i(e-1)}{L_i})$, we have:

$$
\begin{aligned}
-G_i'(u) + G_i(u) &= -G_i'(u) + \gamma_i G_i(u) - (\gamma_i - 1)G_i(u)\\
&= \frac{L_i\gamma_i}{e-1}u - (\gamma_i - 1)G_i(u).
\end{aligned}
\tag{14}
$$

Combining (10), (12), (13) and (14) along with $P_{\text{gseq}} \geq -\frac{Lm\lambda^2}{K}$, we obtain:

$$\begin{aligned}
&\mathrm{D}^* - \max_{i\in[n]} \gamma_i \frac{Lm\lambda^2}{K} \leq \\
&\sum_{t=1}^{m} \sigma_{F_t}\big(d_t(K)\big) - \max_{i\in[n]} \gamma_i \frac{Lm\lambda^2}{K} \\
&- \sum_{i=1}^{n} \big(H_i^*(\nabla H_i(\omega_{im}(K))) + G_i'(\hat{c}_i^T \omega_{im}(K))\big) \\
&\leq \mathrm{ALG} - \sum_{i=1}^{n} \alpha_{H_i} H_i(\omega_{im}(K)) - (\max_{i\in[n]} \gamma_i - 1)\frac{Lm\lambda^2}{K} \\
&+ \sum_{i=1}^{n} \big(\frac{L_i\gamma_i}{e-1} \hat{c}_i^T \omega_{im}(K) - (\max_{i\in[n]} \gamma_i - 1) G_i(\hat{c}_i^T \omega_{im}(K))\big) \\
&\leq \mathrm{ALG} - \sum_{i=1}^{n} \alpha_{H_i} H_i(\omega_{im}(K)) \\
&+ \sum_{i=1}^{n} (\frac{\gamma_i}{e-1} + \max_{i\in[n]} \gamma_i - 1) H_i(\omega_{im}(K)) \\
&\leq \big(1 - \min_{i\in[n]} \alpha_{H_i} + \frac{e}{e-1} \max_{i\in[n]} \gamma_i - 1\big)\mathrm{ALG} \\
&= \big(-\min_{i\in[n]} \alpha_{H_i} + \frac{e}{e-1} \max_{i\in[n]} \gamma_i\big)\mathrm{ALG}.
\end{aligned}$$

Therefore, if $K \to \infty$, the competitive ratio bound is derived as $\frac{\mathrm{ALG}}{\mathrm{D}^*} \geq \frac{1}{-\min_{i\in[n]} \alpha_{H_i} + \frac{e}{e-1} \max_{i\in[n]} \gamma_i}$.

G. Proof of Theorem 4.2

We can use the definition of α to obtain:

$$H_1^*\big(\nabla H_1(\omega_{1m}(K))\big) \geq \alpha_{H_1} H_1(\omega_{1m}(K)). \tag{15}$$

Considering that $G_1'(u) = \ln(\frac{U_1 e}{L_1}) G_1(u)\ ; u \geq \frac{1}{\ln\left(\frac{U_1 e}{L_1}\right)}$, combining (10) and (15) for $n = 1$ along with $P_{\text{gseq}} \geq -\frac{Lm\lambda^2}{K}$, we obtain:

$$\begin{aligned}
&\mathrm{D}^* - \ln(\frac{U_1 e}{L_1})\frac{Lm\lambda^2}{K} \leq \\
&\sum_{t=1}^{m} \sigma_{F_t}\big(d_t(K)\big) - \ln(\frac{U_1 e}{L_1})\frac{Lm\lambda^2}{K} - H_1^*(\nabla H_1(\omega_{1m}(K))) \\
&- G_1'(\hat{c}_1^T \omega_{1m}(K)) \\
&\leq \mathrm{ALG} - \alpha_{H_1} H_1(\omega_{1m}(K)) - \ln(\frac{U_1}{L_1}) G_1(\hat{c}_1^T \omega_{1m}(K)) \\
&- \ln(\frac{U_1}{L_1})\frac{Lm\lambda^2}{K} \\
&\leq \mathrm{ALG} - \alpha_{H_1}\mathrm{ALG} + \ln(\frac{U_1}{L_1})\mathrm{ALG} \\
&= \big(1 - \alpha_{H_1} + \ln(\frac{U_1}{L_1})\big)\mathrm{ALG}.
\end{aligned}$$

Therefore, if $K \to \infty$, the competitive ratio bound is derived as $\frac{1}{1-\alpha_{H_1} + \ln(\frac{U_1}{L_1})}$.