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# On Diverse Bipartite $b$ -Matching

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## Abstract

Bipartite  $b$ -matching, where agents on one side of a market are matched to one or more agents or items on the other, is widely used in many application areas such as healthcare, advertising, and general resource allocation. Traditionally, the primary goal of such models is to maximize a linear function of the constituent matches subject to some constraints. Recent work has studied a new goal of balancing whole-match *diversity* and economic efficiency, where the objective was to maximize coverage over some groups. Basic versions of this problem are solvable in polynomial time. In this work, we provide a generalized version of the problem, where the goal is to simultaneously maximize diversity along several features (e.g., country of citizenship, gender, skills) and show that it is NP-hard. We develop the first combinatorial algorithm that constructs provably-optimal diverse  $b$ -matchings in pseudo-polynomial time. We show that our method guarantees optimal solutions and is faster than state-of-the-art methods for a reviewer assignment application. We conclude with a discussion on key challenges in diverse matching domain.

## 1. Introduction

The bipartite matching problem occurs in many applications such as healthcare, advertising, and general resource allocation. Weighted bipartite  $b$ -matching is a generalization of this problem, where each node on one side of the market can be matched to many items from the other side, and where edges may also have associated real-valued weights. Ex-

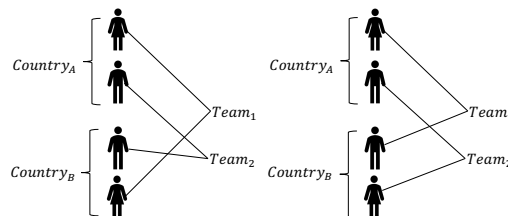


Figure 1. An example of single feature diverse matching (left) versus multi-feature diverse matching (right); here, the matching creates teams with workers from each country and gender.

amples of weighted bipartite  $b$ -matching include assigning children to schools (Drummond et al., 2015; Kurata et al., 2017), reviewers to manuscripts (Charlin & Zemel, 2013; Liu et al., 2014), and donor organs to patients (Dickerson & Sandholm, 2015; Bertsimas et al., 2019).

Ahmed et al. (2017) introduced the notion of *diverse* bipartite  $b$ -matching, where the goal was to simultaneously maximize the “efficiency” of an assignment along with its “diversity.” For example, a firm might want to hire several highly-skilled workers, but if that firm also cares about diversity it may want to ensure that some of those hires occur across marginalized categories of employees. They proposed an objective which combined economic efficiency and diversity demonstrating that, in practice, reducing the efficiency of a matching by small amounts can often lead to significant gains in diversity across a matching (measured by Price of Diversity). However, their formulation was limited to diversity for a single feature and relied on solving a general Mixed-Integer Quadratic Program (MIQP), which is flexible but computationally intractable.

In this work, we generalize the diverse matching problem and introduce matchings where each worker has *multiple* features (e.g., country of origin, gender) and our goal is to form diverse teams with respect to groups in all these features. We found that the problem with a single feature, studied by Ahmed et al. (2017), can be reduced to a minimum quadratic cost maximum flow formulation and solved in polynomial time by an existing algorithm (Minoux, 1986). In contrast, the general case of multiple features is NP-hard. The paper’s main contributions follow:

- We provide the first provably-optimal pseudo-polynomial time algorithm for the diverse bipartite  $b$ -matching w.r.t. multiple features problem.

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- We demonstrate our algorithm’s applicability to paper-reviewer matching. Our algorithm takes less time to converge to an optimal solution than the proposed MIQP approach (using a state-of-the-art commercial solver).

## 2. Related Work

In this section, we will use the example of diverse team formation (for example, in project teams within a company) to provide a concrete example of  $b$ -matching to place prior work in context; however, our proposed approach is generally applicable to any diverse matching problem. We discuss below how diversity among groups of resources is measured and used to form/match teams in literature.

In forming teams, the traditional approach is to use weighted bipartite  $b$ -matching (WBM) methods (Basu Roy et al., 2015). These methods maximize the total weight of the matching while satisfying some constraints. However, there are three major issues with these approaches. First, it assumes that the value provided by a person in a team is always fixed and independent of who else is in the team. This assumption may not hold in many cases. A new team member may provide more added value to the team if she is added to a smaller team compared to the case if she is added to a larger team. This property of diminishing marginal utility can be mathematically captured by submodular functions. Second, existing approaches do not account for diversity *within* a team, where teams with workers from different backgrounds may be desirable. The need for diverse teams is motivated by several studies, including one by McKinsey, which found that firms with better gender and racial diversity have a higher likelihood of increased revenue (Hunt et al., 2015). Finally, every individual may have multiple attributes like gender, race, expertise. A gender balanced team may still have all members from the same race, country or expertise area. In this paper, we address all these issues.

Past researchers have generally measured diversity by defining some notion of coverage via the use of submodular functions, which encode the notion of diminishing returns (Lin & Bilmes, 2012); that is, as one adds items to a set that are similar to previous items, one gains less utility if the existing items in the set already “cover” the characteristics added by that new item. These functions have been shown to achieve top results on common automatic document summarization benchmarks—*e.g.*, at the Document Understanding Conference (Lin & Bilmes, 2012).

Within matching, our work is closest to that of Ahmed et al. (2017), which used a supermodular function to propose a diverse matching optimization method. Other researchers have also approached similar problems, with diversity either as an objective or as a constraint (Agrawal et al., 2018; Lian et al., 2018). Applications of such matching range

from problems in migration (Gölz & Procaccia, 2019), housing (Benabbou et al., 2018), reviewer assignment (Kobren et al., 2019), *etc.* Our goal is to develop an algorithm for finding the optimal assignment which maximizes utility as well as diversity along multiple features as an objective—along with having constraints on workload.

We define a utility function that can be tuned to balance the diversity and total weight of matching. We provide a new algorithm that models the problem using an auxiliary graph and uses a heuristic improvement of the negative cycle detection of Bellman-Ford by Goldberg & Radzik (1993) to find negative cycles and swap people between teams to iteratively reach to the optimal solution for the original problem.

## 3. Preliminaries

In this section, we first define the preliminaries for a diverse matching problem, where workers are to be matched to teams and each team wants workers belonging to a diverse set of categorical features. In our problem, we are given a set of features for the workers. Let  $\mathcal{F} = \{f_1, \dots, f_{|\mathcal{F}|}\}$  denote the feature set for the workers. An example of a feature set could be {country of citizenship, gender}. Each feature  $f_k \in \mathcal{F}$  has one of the values  $\mathcal{F}_k = \{f_{k,1}, \dots, f_{k,|\mathcal{F}_k|}\}$ . Let  $|\mathcal{F}_{k,k'}|$  denote the number of workers having value  $f_{k,k'}$  for feature  $f_k$ . The set of workers is denoted by  $X = \{x_1, \dots, x_n\}$ .  $X$  is partitioned into  $|\mathcal{V}|$  subsets  $V_1, \dots, V_{|\mathcal{V}|}$ , where each subset  $V_j$  corresponds to a feature set  $v_j = \{v_{j,1}, \dots, v_{j,|\mathcal{F}|}\}$ , where  $\forall 1 \leq k \leq |\mathcal{F}|, v_{j,k} \in \mathcal{F}_k$ . We wish to form a set of teams  $\{T_1, \dots, T_t\}$  of the workers where each team  $T_i$  has a demand of  $d_i$ , specifying the number of workers that need to be assigned to it. Each worker can be assigned to at most one team.

The diversity loss of an assignment is denoted by  $D$  and is equal to  $\sum_{k=1}^{|\mathcal{F}|} \lambda_k D_k$ , where  $D_k$  shows the diversity loss w.r.t. feature  $f_k$ , and  $\lambda_k \in \mathcal{Z}^{\geq 0}$  is a constant. Let  $c_{i,k,k'}$  denote the number of workers in  $T_i$  having value  $f_{k,k'} \in \mathcal{F}_k$  for feature  $f_k$ . Then,  $D_k = \sum_{i=1}^t \sum_{k'=1}^{|\mathcal{F}_k|} c_{i,k,k'}^2$ . The cost of assigning each worker having value  $f_{k,k'} \in \mathcal{F}_k$  for feature  $f_k$  to team  $T_i$  is denoted by  $u_{i,k,k'} \in \mathcal{Z}^{\geq 0}$ . We assume all costs are integers. The total cost of an assignment is  $TU = \sum_{k=1}^{|\mathcal{F}|} TU_k$  where  $TU_k = \sum_{i=1}^t \sum_{k'=1}^{|\mathcal{F}_k|} c_{i,k,k'} \cdot u_{i,k,k'}$ .

Our goal is to minimize the objective function which is equal to  $D + \lambda_0 TU$ , where  $\lambda_0 \in \mathcal{Z}^{\geq 0}$ . To understand why minimizing  $D_k$  makes an assignment more diverse w.r.t.  $f_k$ , consider Figure 1. In the left assignment,  $D_2 = 8$ , and in the right assignment  $D_2 = 4$ , and the right assignment is more diverse w.r.t.  $f_2$ , *i.e.* gender. In the full version of this paper we show that this problem is NP-hard. In the next section, we explain our algorithm for finding an optimum

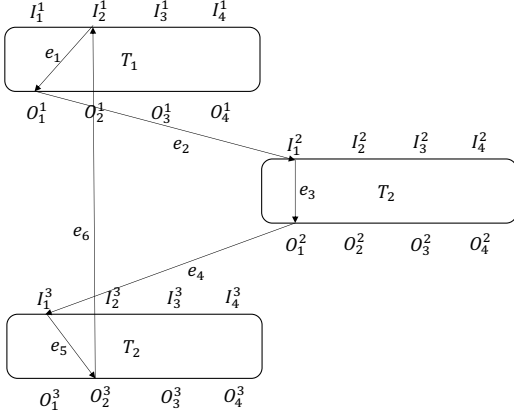


Figure 2. A cycle with weight  $\lambda_0(-u_{3,2,2} + u_{1,2,2} - u_{1,2,1} + u_{3,2,1}) + \lambda_2((c_{3,2,2} - 1)^2 - (c_{3,2,2})^2 + (c_{1,2,2} + 1)^2 - (c_{1,2,2})^2 + (c_{1,2,1} - 1)^2 - c_{1,2,1}^2 + (c_{3,2,1} + 1)^2 - c_{3,2,1}^2)$ .

solution for this NP-hard problem.

#### 4. Negative-Cycle-Detection-based Algorithm

The key insight of our algorithm lies in detecting *negative cycles* in an auxiliary graph representation, which we use to either provide incremental improvements to the incumbent diverse matching or prove that our negative-cycle-detection algorithms have found a globally-optimal matching. First, we build an auxiliary graph  $G'$ . For each team  $T_i \in \{T_0, \dots, T_t\}$ , there is a switch in  $G'$  with  $|\mathcal{V}|$  input ports, and  $|\mathcal{V}|$  output ports. Each port is a node in  $G'$ , and each switch is a directed bipartite graph, with edges going from its input ports (nodes) to its output ports. For each pair of teams  $T_{i_1}$  and  $T_{i_2}$  where  $i_1 \neq i_2$ , and for each feature combination  $v_j$ , there is a directed edge from output port  $O_j^{i_1}$  of switch  $T_{i_1}$  to the input port  $I_j^{i_2}$  of switch  $T_{i_2}$ , and weight of this edge captures the difference in the objective function when a person with feature set  $v_j$  is moved from  $T_{i_1}$  to  $T_{i_2}$ .

In Figure 2, each box is a switch. Inside a switch  $T_i$ , there is a directed edge from each input port to each output port. If the directed edge is connecting two ports such that their corresponding combinations of features do not have the same value for any features, the weight of this edge is equal to zero. Otherwise, per each feature  $f_k$  that has the same value,  $-2\lambda_k$  is added to the weight of this edge. The reason behind assigning these weights to the edges is that if for any value of  $f_k$ ,  $e$  does not change the number of individuals having that value for  $f_k$  in  $T_i$ , e.g. does not change the number of people of any gender in  $T_i$ , then the contribution of  $e$  to  $\Delta(D_k)$  is enforced to be zero.

After constructing the auxiliary graph, we run Algorithm 1 which moves workers from one team to another if it detects a negative cycle. Initially, it takes a solution satisfying demands of all teams as input. When exchanging workers

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#### Algorithm 1 Find optimal diverse $b$ -matching

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while  $\exists$  a negative cycle  $C \in G'$  do
  for  $e \in C$  do
    // Assume edge  $e$  is from output port  $O_j^{i_1}$  of team
     $T_{i_1}$  to input port  $I_j^{i_2}$  of another team  $T_{i_2}$ 
    // Move one worker with feature set  $v_j =$ 
     $\{f_{1,k'_1}, \dots, f_{|\mathcal{F}|,k'_{|\mathcal{F}|}}\}$  from team  $T_{i_1}$  to team  $T_{i_2}$ :
     $\forall k \in \{1, \dots, |\mathcal{F}|\}$ :
       $c_{i_1,k,k'_k} - = 1, c_{i_2,k,k'_k} + = 1$ 
      Update weight of edges of  $G'$  w.r.t to the new values
      of  $c_{i_1,k,k'_k}$ , and  $c_{i_2,k,k'_k}$ 
    end for
  end while

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among the teams along a cycle, the demands of the teams remain fulfilled. In the full version of this paper, we prove Algorithm 1 is guaranteed to find an optimum assignment.

#### 5. Experiments

To demonstrate the effectiveness of the proposed method, we apply it to a dataset of reviewer paper matching. First, we find the optimal solution for multi-feature reviewer paper matching and compare it to the single feature diverse matching method. We also provide the MIQP formulation of the same problem based on literature and show how our algorithm is faster to the Gurobi based MIQP solver.

For the reviewer assignment problem, where each reviewer has multiple features, we want to match each paper with reviewers who are not only from different expertise areas (clusters), but also belong to different genders. We use the multi-aspect review assignment evaluation dataset (Karimzadehgan & Zhai, 2009), a benchmark dataset from UIUC. It contains 73 papers accepted by SIGIR 2007, and 189 prospective reviewers who had published in the main information retrieval conferences. The dataset provides 25 major topics and for each paper in the set, an expert provided 25-dimensional label on that paper based on a set of defined topics. Similarly for the 189 reviewers, a 25-dimensional expertise representation is provided.

To compare our method (Algorithm 1) with a baseline, we formulate a multi-feature MIQP variant of our problem, which is an extension of the single-feature formulation provided in (Ahmed et al., 2017) and is given by:

$$\begin{aligned}
 \min \lambda_0 \sum_{k=1}^{|\mathcal{F}|} \sum_{i=1}^t \sum_{k'=1}^{|\mathcal{F}_k|} u_{i,k,k'} \cdot c_{i,k,k'} + \sum_{k=1}^{|\mathcal{F}|} \lambda_k \sum_{i=1}^t \sum_{k'=1}^{|\mathcal{F}_k|} c_{i,k,k'}^2 \\
 \sum_{k=1}^{|\mathcal{F}|} \sum_{k'=1}^{|\mathcal{F}_k|} c_{i,k,k'} = d_i, \forall 1 \leq i \leq t \\
 \sum_{i=0}^t c_{i,k,k'} = |\mathcal{F}_{k,k'}|, 1 \leq k \leq |\mathcal{F}|, 1 \leq k' \leq |\mathcal{F}_k|
 \end{aligned}$$

To set up the graph for our method, we first cluster the reviewers into 5 clusters based on their topic vectors using spectral clustering. To calculate the relevance of each cluster for any paper, the average cosine similarity of label vectors of reviewers in a cluster is taken. We set the constraints such that each paper matches with exactly 4 reviewers, and no reviewer is allocated more than 1 paper. We augment the dataset for our problem by increasing its size (the number of reviewers are doubled by creating a copy of each reviewer) and adding a augmenting gender data. We added a new feature to each reviewer in this dataset by randomly adding one of two gender labels (Male or Female) to each reviewer. We set  $\lambda_0 = \lambda_1 = \lambda_2 = 1$  for our experiments. Note that by varying these parameters, one can create the Pareto optimal frontier too.

We run the negative cycle detection algorithm, and the MIQP solver using Gurobi to find the optimum solution. On converging to the optimal solution, we find that all 73 papers receive two male reviewers and two female reviewers, which shows that the method was capable of balancing gender diversity. Each paper receives reviewers from four different clusters. If we only optimize for cluster diversity, it is possible that the gender ratio for individual paper gets skewed. When we run the same model with  $\lambda_g = 0$  (no weight to gender diversity), we find that out of 73 papers, 12 papers receive all four reviewers of the same gender and 41 papers receive three reviewers of the same gender. Hence, only 27.3% teams of reviewers are gender balanced for single-feature diverse matching. However, one should note that when we do not keep gender as an objective, the resultant allocation is random and different skewness can be observed in different runs based on the initial solution.

Finally, we compare the timing performance of our algorithm with MIQP by changing the number of papers that need to be reviewed on a Dell XPS 13 laptop with i7 processor. For MIQP, we set a maximum run time of four hours (14400 seconds) for Gurobi solver, at which we report the current best MIQP solution. Table 1 shows that for all cases with the number of papers greater than 13, MIQP does not converge within four hours, while our method finds the optimum solution in lesser time. Interestingly, MIQP current solutions are found to be the same as the optimum solution found by our method, which shows that for this application, MIQP was able to search the solution but it was not able to prove that the solution is optimum. In contrast, our method finds the solution faster as well as guarantees that it is optimum.

## 6. Discussion

In this paper, we proposed the first pseudo-polynomial time algorithms for the NP-hard problem of multi-feature diverse weighted bipartite  $b$ -matching. We propose an algorithm

# Papers	# Reviewers	MIQP Time (s)	Our Method Time (s)
03	378	24.68	0.18
13	378	3979.90	14.84
23	378	14400.00	122.96
33	378	14400.00	400.56
43	378	14400.00	825.95
53	378	14400.00	2837.15
63	378	14400.00	5453.58
73	378	14400.00	11040.55

Table 1. Comparison of MIQP and our method for UIUC reviewer dataset with each paper needing 4 reviewers.

that not only guarantees an optimal solution but also converges faster than a proposed approach using a black-box industrial MIQP solver. We demonstrated our results on a dataset for paper reviewer matching. Below, we discuss a few open questions which need further attention:

### How to generalize diversity measurement for matching problems?

We use a quadratic function to penalize solutions which are not uniformly distributed across different groups. While the specific choice of function allowed for easy-to-compute metric and formulation of the problem as a mixed integer quadratic problem, there is little understanding of what choice of functions may be most appropriate for capturing diversity in bipartite graphs and how to learn and optimize those functions. While advances in learning submodular functions (Balcan et al., 2016), a mixture of submodular functions (Lin & Bilmes, 2012) and learning DPP kernels (Gillenwater et al., 2014) are a promising direction, more work is needed to understand how these methods can be used as objective functions for matching problems and optimized under knapsack and cover constraints.

This work also assumes that all the features over which diversity is measured are known and these categorical features are independent of each other. However, features may be correlated, which can affect multi-attribute matching results. Our future work will focus on using machine learning methods to identify the key features, which are most important in diversity estimation and extend the formulation to correlated features and features in continuous space.

### How much diversity is sufficient?

The objective function used in this work is a weighted sum of matching quality and total diversity. However, for practitioners deciding to use diverse matching algorithms, deciding the value of these weights is non-trivial. For most applications, it is difficult to understand the trade-off between quality and diversity, let alone decide what numerical value of weights to use for a specific choice of objective functions and weights. Further work is needed on combining user-preference elicitation methods to learn an individual’s price of diversity.

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